

This Slideshow was developed to accompany the textbook

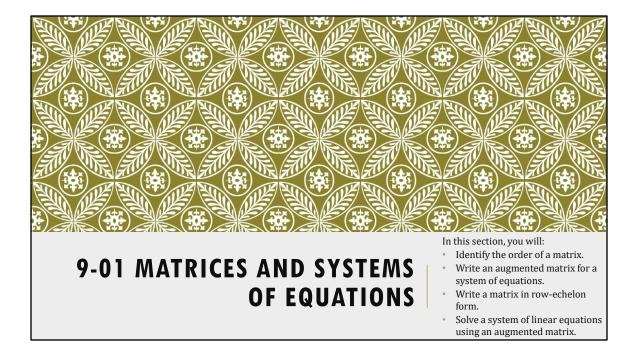
• Precalculus

By Richard Wright

<u>https://www.andrews.edu/~rwright/Precalculus-RLW/Text/TOC.html</u>

Some examples and diagrams are taken from the textbook.

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Matrix

Rectangular array of numbers							
	a ₁₁	<i>a</i> ₁₂	a_{13}	•••	a_{1n}		
_	<i>a</i> ₂₁	$a_{12} \\ a_{22} \\ \vdots$	a_{23}	•••	a_{2n}		
	•	•	•	•.	:		
	a_{m1}	a_{m2}	a_{m3}	•••	a_{mn}		
= (a _{row,c}	olumn					
=]	Each e	entry is	s an ele	emen	ıt		
-Augmented Matrix							
	Two matrices combined together						

Order of matrix • Dimension • Rows × columns

What is the order of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$?

2×3

Elementary Row Operations

Interchange 2 rows

-Multiply a row by a nonzero constant

Add a multiple of a row to another row

Add 2 times 1^{st} row to the 2^{nd} row

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

 $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \end{bmatrix}$

Row-Echelon Form

- All rows consisting entirely of zeros are at bottom
- For other rows, the first nonzero entry is 1
- For successive rows, the leading 1 in the higher row is farther to the left

[1	0	2]	[1	2	3	4]
$\begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$	1	3	0	0	1	4 2 1
Lo	0	0	Lo	0	0	1

Reduced Row-Echelon Form

Columns with leading 1 have 0's as other entries

[1	2	0	0]
$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	0	1	0 0 1
lo	0	0	1

Solve $\begin{cases} x + 3y + 4z = 7\\ 2x + 7y + 5z = 10\\ 3x + 10y + 4z = 27 \end{cases}$

-2×1^{st} add to 2^{nd}	[1 2 3	3 7 10	4 5 4	::	7 10 27
-3×1 st add to 3 rd	$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$	3 1 10	4 -3 4	:	7 -4 27
-1×2^{nd} add to 3^{rd}	$\begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$	3 1 1	4 -3 -8	::	$\begin{bmatrix} 7\\ -4\\ 6 \end{bmatrix}$
-1/5×3 rd	$\begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$	3 1 0	4 -3 -5	::	7 -4 10
2,000	$\begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$	3 1 0 z	4 -3 1 = -	: : 2	7 -4 -2

$$y - 3z = -4 \rightarrow y = -10$$

 $x + 3y + 4z = 7 \rightarrow x = 45$

(45, -10, -2)



9-02 GAUSSIAN ELIMINATION

Gaussian Elimination

 Solving a system of linear equations by putting it into row-echelon form with elementary row operations

Gauss-Jordan Elimination •Solve by putting the system into Reduced row-echelon form

If a row becomes all zeros with final entry not zero = no solution

If a row becomes all zeros = many solutions (do the z = a thing to write the parametric equations of the line of intersection)

9-02 GAUSSIAN ELIMINATION

Solve $\begin{cases} x & -3z = -5\\ 3x + y - 2z = -4\\ 2x + 2y + z = -2 \end{cases}$

-3×1 st add to 2 nd					
-2×1^{st} add to 3^{rd}					-5 11 -2
-2×2^{nd} add to 3^{rd}	$\begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$	0 1 2	-3 7 7	::	$\begin{bmatrix} -5\\11\\8 \end{bmatrix}$
-1/7×3 rd	$\begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$	0 1 0	-3 7 -7	•	-5 11 -14
-7×3^{rd} add to 2^{nd}	$\begin{bmatrix} 1\\ 0\\ 0 \end{bmatrix}$	0 1 0	-3 7 1	::	$\begin{bmatrix} -5\\11\\2 \end{bmatrix}$

	[1	0	-3	÷	-5]
	0	1	0	÷	-3
3×3 rd add to 1 st	Lo	0	1	:	-5 -3 2
	<u>[</u> 1	0	0	:	ן 1
	0	1	0	÷	$\begin{bmatrix} 1\\ -3\\ 2 \end{bmatrix}$
	Lo	0	1	÷	2
(1, -3, 2)					

9-02 GAUSSIAN ELIMINATION

Solve
$$\begin{cases} x + y + 5z = -3 \\ -x - 2y - 8z = 5 \\ -x - 2z = 1 \end{cases}$$

1 st add to 2 nd	$\begin{bmatrix} 1 & 1 & 5 & \vdots & -3 \\ -1 & -2 & -8 & \vdots & 5 \\ -1 & 0 & -2 & \vdots & 1 \end{bmatrix}$
1 st add to 3 rd	$\begin{bmatrix} 1 & 1 & 5 & \vdots & -3 \\ 0 & -1 & -3 & \vdots & 2 \\ -1 & 0 & -2 & \vdots & 1 \end{bmatrix}$
2 nd add to 3 rd	$\begin{bmatrix} 1 & 1 & 5 & \vdots & -3 \\ 0 & -1 & -3 & \vdots & 2 \\ 0 & 1 & 3 & \vdots & -2 \end{bmatrix}$
Many solutions	$\begin{bmatrix} 1 & 1 & 5 & \vdots & -3 \\ 0 & -1 & -3 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$
-1×2 nd	$\begin{bmatrix} 1 & 1 & 5 & \vdots & -3 \\ 0 & 1 & 3 & \vdots & -2 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

 -1×2^{nd} add to 1^{st}

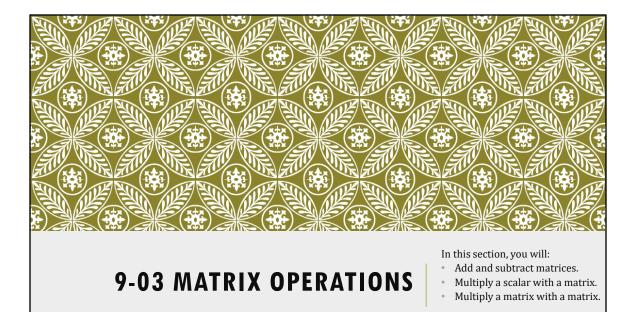
$$\begin{bmatrix} 1 & 0 & 2 & \vdots & -1 \\ 0 & 1 & 3 & \vdots & -2 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$z = a$$

$$y + 3z = -2 \rightarrow y = -2 - 3a$$

$$x + 2z = -1 \rightarrow x = -1 - 2a$$

(-1-2a, -2-3a, a)



Matrix addition and subtraction

- Both matrices must have same order
- Add or subtract corresponding elements

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -4 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -2 & -3 \\ -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 3+0 & 1+(-1) \\ 0+(-2) & 2+(-3) \\ -4+(-4) & -1+(-5) \end{bmatrix}$$
$$\begin{bmatrix} 3 & 0 \\ -2 & -1 \\ -8 & -6 \end{bmatrix}$$

Scalar multiplication • Multiply a matrix with a number

- Distribute

 $3\begin{bmatrix}1&2&3\\0&-1&-2\end{bmatrix}$

 $\begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 & 3 \cdot 3 \\ 3 \cdot 0 & 3 \cdot -1 & 3 \cdot -2 \end{bmatrix} \\ \begin{bmatrix} 3 & 6 & 9 \\ 0 & -3 & -6 \end{bmatrix}$

Matrix multiplication

•Number of columns in 1st = number of rows in 2nd

 $(m \times n) \cdot (n \times p)$

•Order of product $m \times p$

•Order is important

NO COMMUTATIVE PROPERTY!!!!!

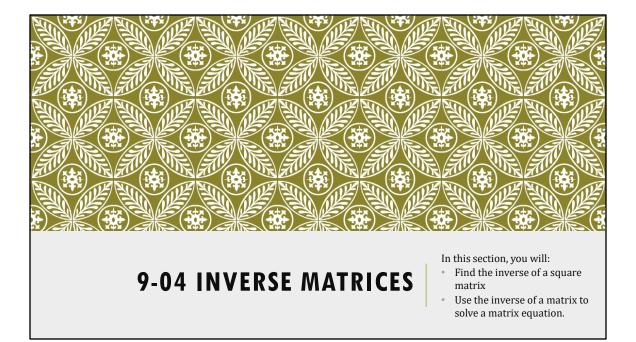


$$\begin{bmatrix} 2 & -1 & 7 \\ 0 & 6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \cdot 0 + (-1)(-2) + 7 \cdot 3 \\ 0 \cdot 0 + 6 \cdot (-2) + (-3) \cdot 3 \end{bmatrix}$$
$$\begin{bmatrix} 23 \\ -21 \end{bmatrix}$$

 $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ -2 & 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2(-1) + 0(-2) & 2(0) + 0(1) & 2(4) + 0(2) \\ 1(-1) + 3(-2) & 1(0) + 3(1) & 1(4) + 3(2) \end{bmatrix}$$
$$\begin{bmatrix} -2 & 0 & 8 \\ -7 & 3 & 10 \end{bmatrix}$$



Identity Matrix (I) $-A \cdot I = A$	$I = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	0 1		
• $A \cdot A^{-1} = I$ • Both A and A ⁻¹ must be square	$I = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	0 1 0	0 0 1	
both A and A must be square	OR $I = \begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}$	0 1 0	0 0 1	0 0 0
	LU	0	U	T]

Inverse of 2×2 • If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ Find the inverse of $\begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$

$$\frac{1}{1(4) - (-2)(0)} \begin{bmatrix} 4 & -0 \\ 2 & 1 \end{bmatrix}$$
$$\frac{1}{4} \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 1/2 & 1/4 \end{bmatrix}$$

Find other inverses

-Augment the matrix with the identity matrix

• Use Gauss-Jordan elimination to turn the original matrix into the identity matrix

$$[A:I] \to [I:A^{-1}]$$

Find the inverse of
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ -3 & 4 & -4 \end{bmatrix}$$

Augment with identity

	α
$\begin{bmatrix} 1 & 2 & 3 & \vdots & 1 & 0 \\ 0 & -1 & -2 & \vdots & 0 & 1 \\ -3 & 4 & -4 & \vdots & 0 & 0 \end{bmatrix}$	0
$L-3 4 -4 \vdots 0 0$	1
3×1 st add to 3 rd	o-
$\begin{bmatrix} 1 & 2 & 3 & \vdots & 1 & 0 \\ 0 & -1 & -2 & \vdots & 0 & 1 \end{bmatrix}$	0
$\begin{bmatrix} 1 & 2 & 3 & \vdots & 1 & 0 \\ 0 & -1 & -2 & \vdots & 0 & 1 \\ 0 & 10 & 5 & \vdots & 3 & 0 \end{bmatrix}$	0
LO 10 5 : 3 0	1
10×2 nd add to 3 rd	
$\begin{bmatrix} 1 & 2 & 3 & \vdots & 1 & 0 \\ 0 & -1 & -2 & \vdots & 0 & 1 \\ 0 & 0 & -15 & \vdots & 3 & 10 \end{bmatrix}$	$\begin{bmatrix} 0\\0\\1 \end{bmatrix}$
0 -1 -2 = 0 1	0
$l_0 0 -15 \vdots 3 10$	1
-2×3 rd add to 15×2 nd	
$\begin{bmatrix} 1 & 2 & 3 & \vdots & 1 & 0 \\ 0 & -15 & 0 & \vdots & -6 & -5 \\ 0 & 0 & -15 & \vdots & 3 & 10 \end{bmatrix}$	0
$0 - 15 0 \vdots -6 -5$	-2 1
$l_0 0 -15 \vdots 3 10$	1
3 rd add to 5×1 st	
[5 10 0 ÷ 8 10	1
$0 -15 0 \vdots -6 -5$	-2
$l_0 0 -15 \vdots 3 10$	1

 $2 \times 2^{nd} \text{ add to } 3 \times 1^{st}$ $\begin{bmatrix} 15 & 0 & 0 & \vdots & 12 & 20 & -1 \\ 0 & -15 & 0 & \vdots & -6 & -5 & -2 \\ 0 & 0 & -15 & \vdots & 3 & 10 & 1 \end{bmatrix}$ $1/15 \times 1^{st}, -1/15 \times 2^{nd}, -1/15 \times 3^{rd}$ $\begin{bmatrix} 1 & 0 & 0 & \vdots & 4/5 & 4/3 & -1/15 \\ 0 & 1 & 0 & \vdots & 2/5 & 1/3 & 2/15 \\ 0 & 0 & 1 & \vdots & -1/5 & -2/3 & -1/15 \end{bmatrix}$ $\begin{bmatrix} 4 & 4 & -1 \\ 5 & 3 & 15 \\ 2 & 1 & 2 \\ 5 & 3 & 15 \\ -\frac{1}{5} & -\frac{2}{3} & -\frac{1}{15} \end{bmatrix}$

Use an inverse to solve system of equations Write system as matrices AX = B (coefficients \cdot variables = constants) $A^{-1}AX = A^{-1}B$ $IX = A^{-1}B$ $X = A^{-1}B$

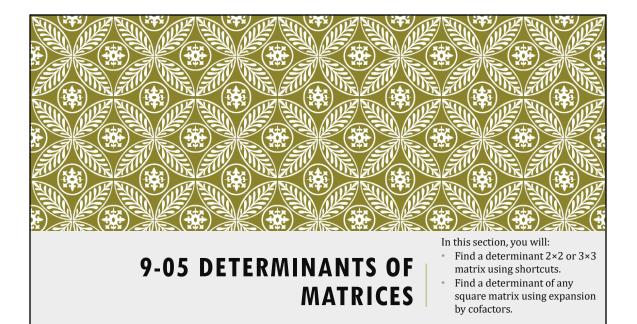
Solve by multiplying the inverse of the coefficients with the constants

Solve $\begin{cases} 2x + 3y = 0\\ x - 4y = 7 \end{cases}$

Find income of coefficients	$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$
Find inverse of coefficients	$\begin{bmatrix} 2 & 3 & \vdots & 1 & 0 \\ 1 & -4 & \vdots & 0 & 1 \end{bmatrix}$
1 st add to -2×3 rd	$\begin{bmatrix} 2 & 3 & \vdots & 1 & 0 \\ 0 & 11 & \vdots & 1 & -2 \end{bmatrix}$
-3×2 nd add to 11×1 st	$\begin{bmatrix} 10 & 11 & : & 1 & -2 \end{bmatrix}$ $\begin{bmatrix} 22 & 0 & : & 8 & 6 \\ 0 & 11 & : & 1 & -2 \end{bmatrix}$
1/22×1 st , 1/11×2 nd	-0 11 , 1 2-
Multiply inverse with constant	$\begin{bmatrix} 1 & 0 & \vdots & 4/11 & 3/11 \\ 0 & 1 & \vdots & 1/11 & -2/11 \end{bmatrix}$ nts
	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ \frac{1}{11} & -\frac{2}{11} \end{bmatrix} \begin{bmatrix} 0 \\ 7 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{21}{11} \\ -\frac{14}{11} \end{bmatrix}$$

(21/11, -14/11)



9-05 DETERMINANTS OF MATRICES

Determinant is a real number associated with a square matrix

Find $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$$2 \times 2$$

• If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then
• det $(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
= $ad - bc$

Down product – up product

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 3(2) = 4 - 6 = -2$$

9-05 DETERMINANTS OF MATRICES

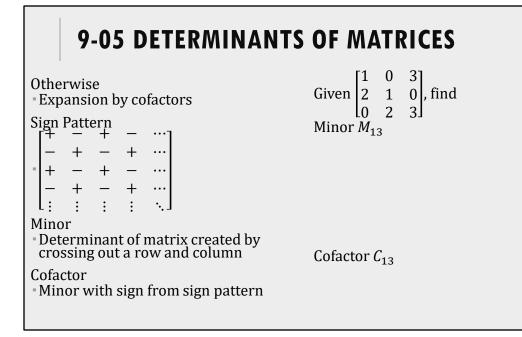
3×3

Copy 1st two columns after matrix + products of downs – products of ups

2 3 1 56

4 7 8 9

> $\begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \end{bmatrix}$ 7 8 9 7 8 $= 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 7 \cdot 5 \cdot 3 - 8 \cdot 6 \cdot 1 - 9 \cdot 4 \cdot 2$ = 45 + 84 + 96 - 105 - 48 - 72= 0



Minor
$$M_{13} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

 $\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$
 $= 4 - 0 = 4$
Cofactor $C_{13} = +(4) = 4$

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9-05 DETERMINANTS OF MATRICES

Find $\begin{vmatrix} -1 & 0 & 4 \\ 3 & -2 & 0 \\ 1 & -1 & 1 \end{vmatrix}$

Pick a row or column with 0's like 1st row Find all the cofactors of 1st row

$$\begin{vmatrix} -1 & 0 & 4 \\ 3 & -2 & 0 \\ 1 & -1 & 1 \end{vmatrix}$$

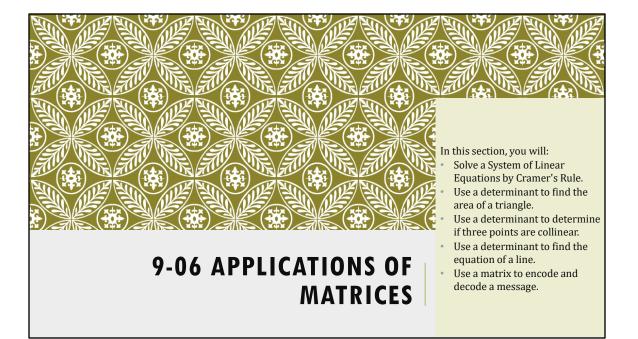
= +(-1) \cdot \bigg|_{-1}^{-2} & 0 \\ -1 & 1 \end{vmatrix} - 0 \cdot \bigg|_{1}^{3} & 0 \\ 1 & 1 \end{vmatrix} + 4 \bigg|_{1}^{3} & -2 \\ 1 & -1 \end{vmatrix}
= -1(-2 - 0) - 0(\bigg) + 4(-3 - (-2))
= 2 + (-4)
= -2

9-05 DETERMINANTS OF MATRICES

Find $\begin{vmatrix} -2 & 4 & 0 & 5 \\ 0 & 2 & -1 & 0 \\ 3 & 1 & -4 & -1 \\ -5 & 0 & -2 & 3 \end{vmatrix}$

Pick 2nd row

$$\begin{array}{c|cccc} -0 & | + 2 & | -2 & 0 & 5 \\ 3 & -4 & -1 \\ -5 & -2 & 3 \\ \end{array} \begin{vmatrix} -2 & 4 & 5 \\ 3 & 1 & -1 \\ -5 & 0 & 3 \\ \end{vmatrix} + 0 | | \\ -5 & 0 & 3 \\ \end{vmatrix} + 0 | | \\ = -0() + 2(24 + 0 + (-30) - 100 - (-4) - 0) \\ + 1(-6 + 20 + 0 - (-25) - 0 - 36) + 0() \\ = 2(-102) + 1(3) \\ = -201 \end{array}$$



Cramer's Rule

Used to solve systems of equations

$$x_1 = \frac{|A_1|}{|A|}$$
 $x_2 = \frac{|A_2|}{|A|}$

A = coefficient matrix

 $-A_n$ = coefficient matrix with column n replaced with constants

If |A| = 0, then no solution or many solutions

Use Cramer's Rule

2x + y + z = 6-x - y + 3z = 1y - 2z = -3

$$x = \frac{\begin{vmatrix} 6 & 1 & 1 & | & 6 & 1 \\ 1 & -1 & 3 & | & 1 & -1 \\ -3 & 1 & -2 & -3 & 1 \\ \hline -3 & 1 & -2 & -3 & 1 \\ \hline -3 & 1 & -2 & -3 & 1 \\ \hline -1 & -1 & 3 & | & -1 & -1 \\ 0 & 1 & -2 & | & 0 & 1 \\ \end{vmatrix}}{y = \frac{\begin{vmatrix} 2 & 6 & 1 & | & 2 & 6 \\ -1 & 1 & 3 & | & -1 & 1 \\ \hline 0 & -3 & -2 & | & 0 & -3 \\ \hline -5 & -5 & -5 & -5 \\ \hline -5 & -5 & -5 & -5 \\ \end{vmatrix}}{= \frac{-4 + 0 + 3 - 0 - (-18) - 12}{-5} = \frac{5}{-5} = -1$$

$$z = \frac{\begin{vmatrix} 2 & 1 & 6 & | & 2 & 1 \\ -1 & -1 & 1 & | & -1 & -1 \\ 0 & 1 & -3 & 0 & 1 \\ \hline -5 & -5 & -5 & -5 \\ \hline -5 & -5 & -5$$

Area of triangle with vertices (x_1, y_1) , Find the area of triangle with vertices (x_2, y_2) , (x_3, y_3) (-3, 1), (2, 4), (5, -3)

$$Area = \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$Area = \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$
$$Area = \pm \frac{1}{2} \begin{vmatrix} -3 & 1 & 1 \\ 2 & 4 & 1 \\ 5 & -3 & 1 \end{vmatrix} \begin{vmatrix} -3 & 1 \\ 2 & 4 \\ 5 & -3 \end{vmatrix}$$
$$= \pm \frac{1}{2} ((-12 + 5 + (-6) - 20 - 9 - 2))$$
$$= \pm \frac{1}{2} (-44)$$
$$= 22$$

Lines in a Plane $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$, then the points are collinear • Find equation of line given 2 points (x_1, y_1) and (x_2, y_2) $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ Find the equation of the line passing through (-2, 9) and (3, -1)

 $0 = \begin{vmatrix} x & y & 1 \\ -2 & 9 & 1 \\ 3 & -1 & 1 \end{vmatrix} \begin{vmatrix} x & y \\ -2 & 9 \\ 3 & -1 & 1 \end{vmatrix} \begin{vmatrix} -2 & 9 \\ 3 & -1 \\ 3 & -1 \end{vmatrix}$ 9x + 3y + 2 - 27 - (-x) - (-2y) = 010x + 5y - 25 = 02x + y - 5 = 0

9-06 APPLICATIONS OF MATRICES								
Hill Cypher Encoding a Message	_=0	I = 9	R = 18					
 Convert the message into numbers Choose a square encoding matrix. 	A = 1	J = 10	S = 19					
3. Group the message numbers into matrices of 1 row	B = 2	K = 11	T = 20					
and the same number of columns as the encoding matrix.	C = 3	L = 12	U = 21					
4. Multiply the letter matrices with the encoding	D = 4	M = 13	V = 22					
matrix.	E = 5	N = 14	W = 23					
5. The encoded message is the list of numbers produced.	F = 6	0 = 15	X = 24					
• Decode by using inverse of encoding matrix		P = 16	Y = 25					
	H = 8	Q = 17	Z = 26					

Encode LUNCH using $\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$

Letters become 12, 21, 14, 3, 8, 0

$$\begin{bmatrix} 12 & 21 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 54 & -63 \end{bmatrix}$$
$$\begin{bmatrix} 14 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 20 & -9 \end{bmatrix}$$
$$\begin{bmatrix} 8 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \end{bmatrix}$$

Message: 24, -63, 20, -9, 8, 0